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# Reduced-Order Aeroservoelastic Model with an Unsteady Aerodynamic Eigen Formulation

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## Introduction

A DIFFICULTY in using modern control theory for the design of aeroelastic control systems is the requirement of transforming the unsteady aerodynamic forces, normally provided in the frequency domain, into the time domain. The usual procedure is to approximate the unsteady aerodynamic force as a rational function in the Laplace domain.

Recently, a new formulation called eigenanalysis of unsteady aerodynamics has provided a direct representation of the unsteady aerodynamic loads in the time domain.[1] The eigenmethods suggested so far rely on the discrete-time domain formulation that may result in reduced-order models with complex coefficients. For the designs of closed-loop aeroelastic systems, however, it is preferable to have aerodynamic models with real coefficients in the continuous-time domain. The purpose of this study is to construct a new aeroservoelastic model using a two-dimensional, incompressible aerodynamic eigenformulation in the continuous-time domain. The work is based on the discrete-time domain eigen model by Hall.[2] To obtain a reduced-order model with real coefficients, a modal decomposition technique that transforms the aerodynamic matrix into a canonical modal form is introduced. A simple typical section model with two strain-actuated control inputs is used for an aeroservoelastic design.

## Unsteady Aerodynamic Modeling

We divide the bound vortex representing the airfoil into $M$ elements and divide the free vortex representing the traveling wake into $(N - M)$ elements. The total number of vortex elements is $N$.

Given an $(M \times 1)$ bound vortex vector $\boldsymbol{\Gamma}_1$ and an $[(N - M) \times 1]$ wake vortex vector $\boldsymbol{\Gamma}_2$, the downwash in the discrete-space domain is expressed as

$$\boldsymbol{W}_{\frac{3}{4}} = [\boldsymbol{K}_1 \quad \boldsymbol{K}_2]\begin{Bmatrix}\boldsymbol{\Gamma}_1\\ \boldsymbol{\Gamma}_2\end{Bmatrix} \tag{1}$$

where $\boldsymbol{W}_{3/4}$ is an $(M \times 1)$ vector containing downwashes at three-quarter points of the airfoil elements and $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are the two-dimensional kernel function matrices.

For conservation of vorticity in the discrete-space domain, it is required that

$$-\boldsymbol{S}_1\dot{\boldsymbol{\Gamma}}_1 = (\Gamma_{M+1}/\Delta x)U \tag{2}$$

with $\boldsymbol{S}_1 \equiv \lfloor 1 \ 1 \ \ldots \ 1 \rfloor (1 \times M)$. For the special case of $\Delta x = U\Delta t$, we can write

$$\Gamma^{n+1}_{M+1} + \Gamma^{n}_{M+1} = -2\sum^{M}_{i}(\Gamma_i^{n+1} - \Gamma_i^{n}) \tag{3}$$

For convection of free wakes, we have

$$\dot{\Gamma}_i = -U\frac{\partial \Gamma_i}{\partial x} \qquad (i = M+2, \ldots, N) \tag{4}$$

The free wake convection is conveniently described in the discrete-time domain as

$$\begin{aligned} \Gamma_i^{n+1} &= \Gamma_{i-1}^{n} \qquad (i = M+2, \ldots, N-1)\\ \Gamma_N^{n+1} &= w\Gamma_N^{n} + \Gamma_{N-1}^{n} \qquad (0.95 < w < 1) \end{aligned} \tag{5}$$

The weighting factor $w$ in Eq. (5) is introduced to prevent sudden change in the induced downwash due to the finite length of the wake vortex sheet.[2]

By putting Eqs. (1), (2), and (4) together, one can write the continuous model in matrix form:

$$\boldsymbol{A}_a\dot{\boldsymbol{\Gamma}}_2 = (U/\Delta x)\boldsymbol{B}_a\boldsymbol{\Gamma}_2 + \boldsymbol{C}_a\dot{\boldsymbol{W}}_{\frac{3}{4}} \tag{6}$$

The corresponding discrete model can be written in a fashion similar to but simpler than that of Hall[2] as

$$\boldsymbol{A}_{ad}\boldsymbol{\Gamma}_2^{n+1} = \boldsymbol{B}_{ad}\boldsymbol{\Gamma}_2^{n} + \boldsymbol{C}_{ad}\dot{\boldsymbol{W}}^{n+1}_{\frac{3}{4}} \tag{7}$$

where the time step in this discrete model is $\Delta t = \Delta x/U$. In the preceding equations, all matrices are known except for $\boldsymbol{B}_a$. It can be obtained by integrating Eq. (6) from $t = t^n$ to $t = t^{n+1}$ and comparing the homogeneous part with that of Eq. (7):

$$\boldsymbol{B}_a = \boldsymbol{A}_a \ln(\boldsymbol{A}_{ad}^{-1}\boldsymbol{B}_{ad}) \tag{8}$$

## Reduced-Order Aeroservoelastic Model

To obtain a reduced-order model with real coefficients, we first transform the aerodynamic matrix into a canonical modal form,

$$\boldsymbol{A}_{\text{can}} = \boldsymbol{Q}^T(\boldsymbol{A}_a^{-1}\boldsymbol{B}_a)\boldsymbol{P}, \qquad \boldsymbol{Q}^T\boldsymbol{P} = \boldsymbol{I}_{N-M} \tag{9}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}^T$, respectively, contain the right and left eigenvectors of the system. In this form, pairs of complex conjugate eigenvalues appear in $(2 \times 2)$ blocks along the diagonal, and the real eigenvalues appear on the diagonal. We next decompose the unsteady vortex into

$$\bar{\boldsymbol{\Gamma}}_2 \simeq \boldsymbol{P}_R\boldsymbol{q} + \bar{\boldsymbol{\Gamma}}_s \tag{10}$$

where $\boldsymbol{P}_R$ contains the $N_R$ columns of $\boldsymbol{P}$ corresponding to the selected $N_R$ eigenvalues, $\boldsymbol{q}$ is the generalized coordinate vector, and $\bar{\boldsymbol{\Gamma}}_s$ is a static correction.

Finally, for a typical section undergoing plunging $\bar{h}$ and pitching motion $\alpha$ with bending and torsion strain actuation $u_h$ and $u_\alpha$, respectively, one can write nondimensional aeroservoelastic equations of motion in the following form:

$$\boldsymbol{A}\dot{\boldsymbol{X}} = \boldsymbol{B}\boldsymbol{X} + \boldsymbol{C}\boldsymbol{u} \tag{11}$$

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where $\boldsymbol{X} \equiv \lfloor \boldsymbol{q}\ \bar{h}\ \alpha\ \dot{\bar{h}}\ \dot{\alpha} \rfloor^T$ and $\boldsymbol{u} \equiv \lfloor u_h\ u_\alpha \rfloor^T$. The dimension of the system is $(N_R + 4)$.

## Illustration

Table 1 shows the structural and geometric parameters of the typical section model used. For this study, 10 and 100 vortex elements are used to model the airfoil and the wake, respectively.

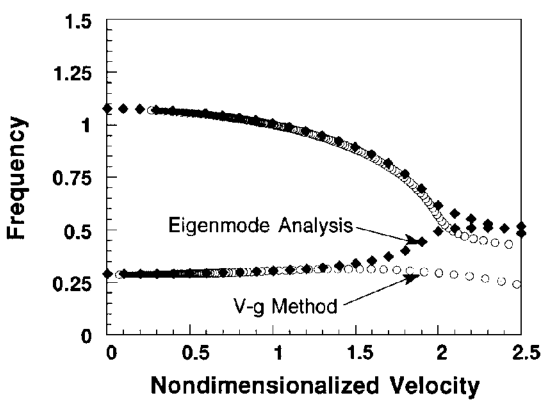
First, the eigenmode information is used to compute the open-loop flutter speed. For model reduction, a total of 20 aerodynamic eigenmodes nearest to the origin is selected. Figures 1a and 1b show the nondimensional frequency ($\omega/\omega_\alpha$) and damping of the reduced-order model against the reduced velocity $U_\alpha$ ($= U/\omega_\alpha b$). The flutter occurs at 2.01, which is very close to Hall's result. The flutter speed calculated using the $V$-$g$ method is about 2.0. The predicted divergence speed from the $V$-$g$ method is about 2.5 and is the same as the current result.

The control system for flutter suppression is designed with two strain actuators by the linear quadratic regulator with the output feedback where the optimal control gain matrix is obtained by solving a set of three coupled nonlinear algebraic matrix equations simultaneously.[3] The design airspeed is specified at 2.1, which is higher than the open-loop flutter speed. Figures 2 and 3 show the closed-loop root loci for two different values, $\bar{\rho} = 0.01$ and 10.0. The control system designed with $\bar{\rho} = 0.01$ is more stable. The unstable airspeed of this system is about 3.4 (divergence), which is higher than that of the system designed with $\bar{\rho} = 10.0$. It can be seen that the control performance is better when the weighting parameter is smaller.

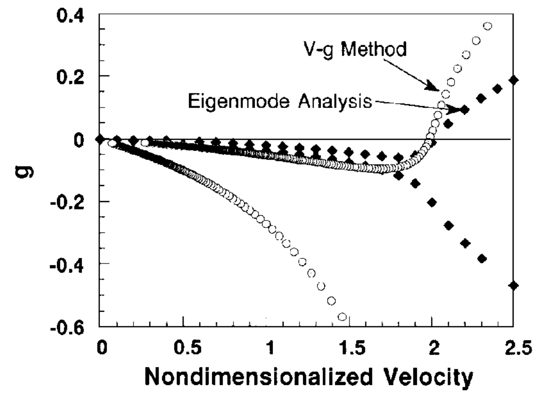
Finally, a comparison of CPU times for a typical flutter calculation is in order. On the Power MAC 6100/66 system, the full-order model required 3 min and 55 s, whereas the reduced-order model required only 39 s.

**Table 1 Geometric properties of the typical section**

| Parameter | Values |
|---|---|
| $x_\alpha/b$ | 0.2 |
| $\omega_h/\omega_\alpha$ | 0.3 |
| $r_\alpha^2/b^2$ | 0.25 |
| $\mu$ | 20.0 |
| $e/b$ | $-0.1$ |
| State weight $Q_{hh}$ | $1/0.406^2$ |
| State weight $Q_{\alpha\alpha}$ | $1/0.282^2$ |
| Control weight $R_{hh}$ | $1/0.0429^2$ |
| Control weight $R_{\alpha\alpha}$ | $1/0.0215^2$ |




a) Frequency




b) Damping plots

**Fig. 1 Typical section with the variation of the reduced velocity.**

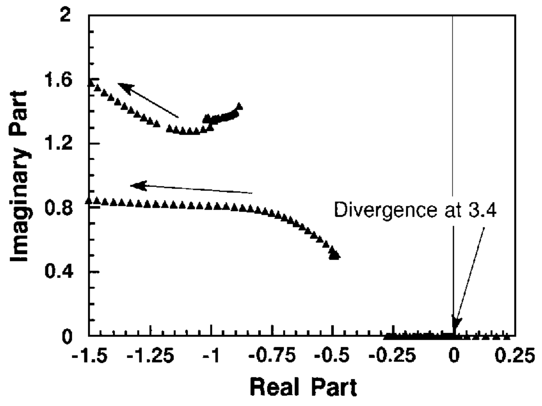



**Fig. 2 Closed-loop root loci of the model when the weighting parameter $\bar{\rho}$ = 0.01.**

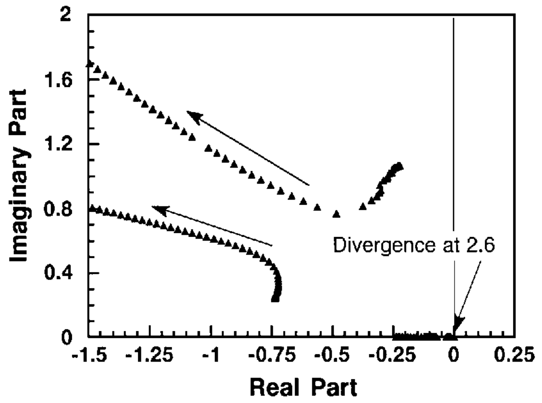



**Fig. 3 Closed-loop root loci of the model when the weighting parameter $\bar{\rho}$ = 10.0.**

## Concluding Remarks

In this study, a new reduced-order aeroservoelastic model with real coefficients has been developed using a two-dimensional, incompressible eigenformulation in the continuous-time domain. A simple two-dimensional typical section with plunging and pitching degrees of freedom, along with two strain-actuated control inputs, is used for an aeroservoelastic design. It is expected that when the current model is extended to a three-dimensional wing model, there will be a substantial saving in CPU time compared with the existing methods, such as the frequency domain analysis with rational function approximation.

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